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The onset of steady Bénard-Marangoni convection in a two-layer system of conducting fluid in the presence of a uniform magnetic field

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Abstract. The onset of steady Bénard-Marangoni convection in two horizontal liquid layers of electrically conducting immiscible fluids subjected to a uniform vertical magnetic field and temperature gradient is analysed by means of a combination of analytical and numerical techniques. The free surface can be either deformable or nondeformable and the interface between the fluids is always assumed to be flat. The effect of the lower layer on the critical values of Rayleigh, Marangoni and wave numbers for the onset of steady convection is investigated. When the free surface is nondeformable, the critical parameters for the onset of pure Marangoni convection are increased, whereas for the onset of pure Bénard convection they are decreased compared to the single-layer model. The results for a single-layer and for two-layers are qualitatively similar for Bénard-Marangoni convection when the free surface is deformable. All disturbances can be stabilized with sufficiently strong magnetic field when the free surface is nondeformable. If the free surface is allowed to deform and gravity waves are excluded, then the layers are always unstable to disturbances with sufficiently small wave number with magnetic field. Inclusion of gravity waves has a stabilizing effect on certain disturbances of small wave number in the presence of weak or moderate magnetic field.

Key words: Bénard-Marangoni convection, conducting fluids, stability, two-layer system.

1. Introduction

Buoyancy-driven instability has been well established since the pioneering work of Rayleigh [1], who showed that convection, called Bénard convection, occurs only when the Rayleigh number exceeds a critical value. On the other hand, Pearson [2] showed that surface-tension-gradient effects can also cause convection, usually called Marangoni convection, when the Marangoni number exceeds a critical value.

The combined problem, including both buoyancy and surface-tension effects, was treated by Nield [3] in the the limit of large surface tension who found that for steady convection the two destabilizing agencies reinforce one another and are tightly coupled. Davis and Homsy [4] studied the effect of weak free-surface deformation on critical Rayleigh and Marangoni numbers for the onset of steady convection. They concluded that the presence of a deformable free surface can lead to a stabilization of Bénard convection and the destabilization of Marangoni convection relative to the case of a planar free surface. Takashima [5, 6] investigated the effect of stronger free-surface deformation on the onset of steady and oscillatory Marangoni convection, respectively.

The effect of a uniform vertical magnetic field on pure Bénard convection was described by Chandrasekhar [7] who demonstrated that the presence of the magnetic field increases the critical value of Rayleigh number for the onset of both steady and overstable convection, and so has a stabilizing effect on the layer. Nield [8] analyzed the effect of a magnetic field on both Bénard and Marangoni convection by extending his previous work [3] to include a vertical magnetic field. He found that the effect of increasing the strength of the magnetic field was to monotonically increase the critical values of Rayleigh and Marangoni numbers and hence stabilize the layer. Sarma [9] investigated the effect of magnetic field on the onset of steady Marangoni convection with deformable free surface and Bénard-Marangoni convection for a variety of thermal and magnetic boundary conditions was considered in [10]. Wilson [11] pointed out the incorrectness of the boundary condition at the deformable free surface used by Sarma [9, 10]. Wilson [12] investigated the effect of magnetic field on the onset of steady Bénard-Marangoni convection in a horizontal layer of fluid, when the free surface is deformable with conducting lower rigid boundary condition and concluded that the presence of a magnetic field always has a stabilizing effect on the layer. Treating the Marangoni number as the critical parameter, he showed that, if the free surface is nondeformable, then any particular disturbance can be stabilized with a sufficiently strong magnetic field, but, if the free surface is deformed and gravity waves are excluded, then the layer is always unstable to small wave number disturbances with or without the magnetic field. Including gravity has a stabilizing effect on the long wave-length modes, but not all disturbances can be stabilized, no matter how strong the magnetic field is. Thess and Nitschke [13] investigated the spatial structure of the marginal mode for steady Marangoni convection in the limit of strong magnetic field.

The onset of Marangoni convection including gravity waves (but not buoyancy), in two initially motionless viscous immiscible fluids confined between horizontal isothermal solid surfaces, has been investigated by Smith [14] who found instability for heating vertically above and below. Zeren and Reynolds [15] included buoyancy forces in the analysis of Smith [14] and concluded that stability depends strongly on the ratios of the properties of the fluids, the total depth of the layer and the depth fraction of each fluid. Furthermore, the addition of the stable density gradient increases the critical Marangoni number for moderate and large wave-number disturbances. In all the studies referred to above, a temperature gradient applied perpendicular to the free surface induces the convective motion. On the other hand, Crespo and del Arco *et al.* [16] and Doi and Koster [17] have studied thermocapillary convection due to temperature gradient applied parallel to the interfaces, in two immiscible liquid layers in a rectangular cavity with flat interfaces. They obtained an analytical solution, assuming infinite horizontal layers in order to understand the basic physics of the liquid encapsulation. Recently Biswal and Rao [18] have extended these ideas to deformable interfaces. For the infinite layers considered here, the deformation of the free surface corresponds to nonzero small Crispation number. They concluded that the assumption of flat interface and free surface is good for certain fluids (see also Doi and Koster [17]). Therefore, the assumption of flat interface is reasonable for the case of large interface surface tension, even when the free surface is slightly deformable.

In the present paper a linear stability analysis is carried out for two initially motionless, viscous, immiscible and infinite fluid layers bounded by a horizontal isothermal solid surface below and a free surface above. We mainly assume that both the interface and free surface are flat, but also give some results for the case of a slightly deformable free surface. Here we consider the onset of steady Bénard-Marangoni convection for a system heated from below. Under the application of a uniform vertical magnetic field the influence of Rayleigh number and Chandrasekhar number on the critical Marangoni number is examined. The primary interest is to know how marginal stability characteristics are modified by including Marangoni and Rayleigh numbers simultaneously in the presence of a magnetic field for a two-layer

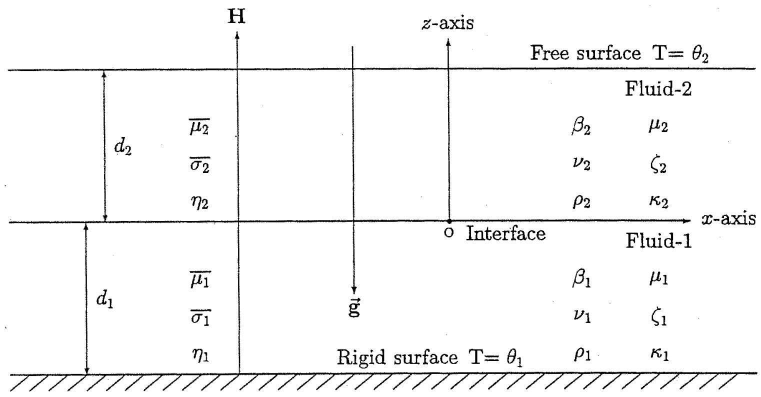


Figure 1. Geometry of the unperturbed state. The y-axis is normal in to the page.

system. The two-layer system introduces a large number of dimensionless parameters, but only a selected and exemplifying set of results are displayed.

2. Mathematical formulation

Consider two unbounded horizontal layers of quiescent fluid, one lying above the other as shown in Figure 1, subject to a uniform vertical magnetic field of strength H in which the lower rigid surface is maintained at constant temperature θ_1 and the free surface, which is bounded with a passive electrically nonconducting gas of negligible density, is maintained at constant temperature θ_2. A Cartesian coordinate system is chosen with x, y-axes in the plane of the interface and z-axis vertically upwards. Fluid 1 is bounded below by $z = -d_1$ and the free surface of fluid 2 which is above the fluid 1, is at $z = d_2$. It is assumed that the interface surface tension is large enough to make the fluid-fluid interface, at $z = 0$, flat when motion occurs, whereas the free surface is deformed and given by $z = d_2 + f(x, y, t)$. Let $\rho_i, \kappa_i, \mu_i, \zeta_i, \nu_i, \bar{\mu}_i, \bar{\sigma}_i$ and $\eta_i = 1/4\pi\bar{\mu}_i\bar{\sigma}_i$ be the densities, the thermal diffusivities, the dynamic viscosities, the thermal conductivities, the kinematic viscosities, the magnetic permeabilities, the finite electrical conductivities and the nonzero electrical resistivities of the fluids respectively. In what follows subscript i takes values 1 and 2 identifying the variables of the lower and the upper layers, respectively. In order to avoid Rayleigh-Taylor instability, we assume $\rho_1 \geqslant \rho_2$. In the initial state, the temperatures, and hence the densities, vary linearly in the z direction and the surface tensions at the free surface and the interface are uniform. The surface tensions may vary along the free surface and the interface, but all other properties are assumed to be uniform in each medium. The surface tensions at the interface and the free surface are given by the linear equations

$$\sigma^I = \sigma_c^I - \sigma_T^I(T - \theta_0) \quad \text{at } z = 0, \tag{1}$$

$$\sigma^F = \sigma_c^F - \sigma_T^F(T - \theta_2) \quad \text{at } z = d_2 + f(x, y, t), \tag{2}$$

respectively, where θ_0 is the interface temperature in the conductive state, the positive constants σ_T^I and σ_T^F are the surface-tension temperature coefficients and, σ_c^I and σ_c^F are the surface tensions at the appropriate reference temperatures. In (1) and (2) and in what follows superscripts I and F indicate the corresponding quantities at the interface and the free surface, respectively.

3. General flow equations and boundary conditions

The complete equations of motion for incompressible, electrically conducting fluids in the presence of a magnetic field subject to the Boussinesq approximation are

$$\frac{\partial \boldsymbol{U}_i}{\partial t} + (\boldsymbol{U}_i.\nabla)\boldsymbol{U}_i = -\frac{\rho_i}{\rho_{ci}}\boldsymbol{g} - \frac{1}{\rho_{ci}}\nabla\Pi_i + \nu_i\nabla^2\boldsymbol{U}_i + \frac{\bar{\mu}_i}{4\pi\rho_{ci}}(\boldsymbol{H}_i.\nabla)\boldsymbol{H}_i, \tag{3}$$

$$\frac{\partial \boldsymbol{H}_i}{\partial t} + (\boldsymbol{U}_i.\nabla)\boldsymbol{H}_i = (\boldsymbol{H}_i.\nabla)\boldsymbol{U}_i + \eta_i\nabla^2\boldsymbol{H}_i, \tag{4}$$

$$\frac{\partial T_i}{\partial t} + \boldsymbol{U}_i.\nabla T_i = \kappa_i\nabla^2 T_i, \tag{5}$$

$$\nabla.\boldsymbol{U}_i = 0, \tag{6}$$

$$\nabla.\boldsymbol{H}_i = 0, \tag{7}$$

where $\boldsymbol{U}_i$ are the fluid velocities, $\boldsymbol{H}_i$ are the magnetic fields, T_i are the temperatures, $\boldsymbol{g} = (0, 0, g)$ is the external gravity field and Π_i are the magnetic pressures, which are defined to be $\Pi_i = P_i + \bar{\mu}_i|\boldsymbol{H}_i|^2/8\pi$, where P_i are the fluid pressures. The derivation of (3) to (7) is given in Chandrasekhar [7, pp. 160–161]. The densities of the fluids are given by

$$\rho_i = \rho_{ci}(1 - \alpha_i(T_i - \theta_i)), \tag{8}$$

where α_i are the constant coefficients of volume expansion and the ρ_{ci} are values of the constant densities at the reference temperatures θ_i.

The external magnetic field in the region below the fluid 1 is give by $\boldsymbol{H}_1^{\mathrm{e}} = \nabla\phi_1$, where $\nabla^2\phi_1 = 0$ and $\phi_1 = 0$ at $z = -\infty$. Similarly, for the region above the free surface $\boldsymbol{H}_2^{\mathrm{e}} = \nabla\phi_2$, where $\nabla^2\phi_2 = 0$ and $\phi_2 = 0$ at $z = \infty$. These boundary conditions apply to nonconducting exteriors, see Chandrasekhar [7, pp. 162–163].

The boundary conditions for the problem are given by

$$\boldsymbol{U}_1 = 0, T_1 = \theta_1, \boldsymbol{H}_1 = (0, 0, H) + \nabla\phi_1 \tag{9}$$

at the lower boundary $z = -d_1$,

$$\boldsymbol{U}_1.\mathbf{n}^I = \boldsymbol{U}_2.\mathbf{n}^I = 0, (\boldsymbol{U}_1 - \boldsymbol{U}_2).\mathbf{t}_x^I = 0, (\boldsymbol{U}_1 - \boldsymbol{U}_2).\mathbf{t}_y^I = 0, \tag{10}$$

$$\zeta_1 T_{1\mathbf{n}^I} = \zeta_2 T_{2\mathbf{n}^I}, T_1 = T_2, \tag{11}$$

$$(S_{jk}^{(1)} - S_{jk}^{(2)})n_k^I n_j^I = 0, (S_{jk}^{(1)} - S_{jk}^{(2)})n_k^I t_{xj}^I = \sigma_{t_x^I}^I, (S_{jk}^{(1)} - S_{jk}^{(2)})\eta_k^I t_{yj}^I = \sigma_{t_y^T}^I, \tag{12}$$

$$\boldsymbol{H}_1 = \boldsymbol{H}_2 \tag{13}$$

at the interface $z = 0$ and

$$\boldsymbol{U}_2.\mathbf{n}^F = \frac{\partial f(x, y, t)}{\partial t}, S_{jk}^{(2)} n_k^F t_{xj}^F = \sigma_{t_x^F}^F, S_{jk}^{(2)} n_k^F t_{yj}^F = \sigma_{t_y^F}^f, \tag{14}$$

$$S_{jk}^{(2)} n_k^F n_j^F = \sigma^F K, \zeta_2 T_{2\mathbf{n}^F} + \gamma(T_2 - \theta_g) = 0, \tag{15}$$

$$\boldsymbol{H}_2 = (0, 0, H) + \nabla\phi_2 \tag{16}$$

at the free surface $z = d_2 + f(x, y, t)$.

In (15) the symbols γ, θ_g and K are the heat-transfer coefficient between the free surface and the gas lying above, the ambient temperature of the above lying gas and the free surface curvature, respectively. The components of the stress tensors $S_{jk}^{(i)}$ of the fluids, in the usual tensor notation, are defined by

$$S_{jk}^{(i)} = -P_i\delta_{jk} + 2\mu_i\varepsilon_{jk}^{(i)}, 2\varepsilon_{jk}^{(i)} = (u_{j,k}^{(i)} + u_{k,j}^{(i)}), \tag{17}$$

where δ_{jk} is Kronecker delta. Here, we have used superscript (i), $i = 1, 2$ indicating the fluid 1 and 2. The subscripts $\mathbf{n}^I$, $\mathbf{t}_x^I$, $\mathbf{t}_y^I$, $\mathbf{n}^F$, $\mathbf{t}_x^F$ and $\mathbf{t}_y^F$ represent the normal and the tangential derivatives at the interface and the free surface. Derivation of the above boundary conditions is given in Smith [14] and Chandrasekhar [7, pp. 162–163]. Equation (10) and the first condition of (14) are the kinematic boundary conditions at the interface and the free surface, respectively. Note that, since the fluids have nonzero electrical resistivity, $\eta_i \neq 0$, all the components of the magnetic field are continuous across the interface and the free surface and, consequently, there are no net magnetic stresses at the interface and the free surface. The balance of stress at the interface and the free surface in the direction are given by the first conditions of (12) and (15), respectively. The first condition of (12) implies that the P_i are continuous across the interface. The jump in the normal stress across the free surface is balanced by the surface tension times the curvature as seen from the first condition of (15). Similarly, the tangential stress balance at both the interface and the free surface in the x and y directions are given by the second and the third conditions of (12) and (14), respectively.

4. Basic state solutions

In the basic state the heat flow is due only to conduction and the fluids are at rest. In this case $\boldsymbol{U}_i = 0$, the magnetic field is uniform, $\boldsymbol{H} = (0, 0, H)$, $\phi_i = 0$, the free surface is flat, $f(x, y, t) = 0$ and there is a uniform adverse temperature gradient β_i across each layer, so $T_i = \theta_0 + \beta_i z$, where

$$\theta_0 = \frac{\theta_1\zeta_1 d_2 + \theta_2\zeta_2 d_1}{\zeta_1 d_2 + \zeta_2 d_1}, \qquad \beta_1 = -\frac{(\theta_1 - \theta_2)\zeta_2}{\zeta_1 d_2 + \zeta_2 d_1}, \qquad \beta_2 = -\frac{(\theta - \theta_2)\zeta_1}{\zeta_1 d_2 + \zeta_2 d_1}.$$

Pressures P_1 and P_2, and θ_g are given by

$$P_1 = P_0 + g\rho_{c2}d_2\left(1 + \frac{\alpha_2\beta_2 d_2}{2}\right) - g\rho_{c1}z\left(1 + \frac{\alpha_1\beta_1}{2}(z + 2d_1)\right), \tag{18}$$

$$P_2 = P_0 + g\rho_{c2}(d_2 - z)\left(1 + \frac{\alpha_2\beta_2}{2}(d_2 - z)\right), \tag{19}$$

$$\theta_g = \theta_2 + \frac{\kappa_2\beta_2}{\gamma}, \tag{20}$$

Table 1. Definitions of nondimensional parameters and ratios.

$\mathrm{Ra}_i = -g\alpha_i\beta_i d_2^4/\nu_i\kappa_i$	Rayleigh numbers
$\mathrm{Ma}_1 = -\sigma_T^I\beta_1 d_2^2/\mu_1\kappa_1$	Marangoni number at the interface
$\mathrm{Ma}_2 = -\sigma_T^F\beta_2 d_2^2/\mu_2\kappa_2$	Marangoni number at the free surface
$\mathrm{Pr}_i = \nu_i/\kappa_i$	Prandtl numbers
$\mathrm{Pm}_i = \eta_i/\kappa_i$	Magnetic Prandtl numbers
$\mathrm{Cr} = \mu_2\kappa_2/\sigma_c^F d_2$	Crispation number
$Q_i = \bar{\mu}_i H^2 d_2^2/4\pi\mu_i\eta_i$	Chandrasekhar numbers
$\mathrm{Bi} = \gamma d_2/\zeta_2$	Biot number
$\mathrm{Bo} = \rho_{c2} g d_2^2/\sigma_c^F$	Bond number
$\rho = \rho_{c2}/\rho_{c1}$	density ratio
$\mu = \mu_2/\mu_1$	viscosity ratio
$\zeta = \zeta_2/\zeta_1 (= 1/\beta = \beta_1/\beta_2)$	thermal conductivity ratio
$\kappa = \kappa_2/\kappa_1$	thermal diffusivity ratio
$\eta = \eta_2/\eta_1$	electrical resistivity ratio
$\bar{\mu} = \bar{\mu}_2/\bar{\mu}_1$	magnetic permeability ratio
$d = d_1/d_2$	depth fraction
$\alpha = \alpha_2/\alpha_1$	volume expansion ratio
$\lambda = \sigma_T^I/\sigma_T^F$	surface tension gradient ratio

where P_0 is the constant atmospheric pressure.

5. Linearized normal mode analysis

We introduce nondimensional variables, taking d_2, κ_2/d_2, $H, -\beta_2$, $\rho_{c2}\kappa_2^2/d_2^2$ and d_2^2/κ_2 as the appropriate scales for the unit of length, velocity, magnetic field, temperature gradient, pressure and time, into the governing equations and the boundary conditions. We obtain the 17 nondimensional groups given in Table 1. Note that some of the nondimensional parameters for the fluid 1 and 2 are related through: $Q_1 = (Q_2\mu\eta)/\bar{\mu}$, $\mathrm{Ma}_1 = \lambda\mu\kappa\zeta\,\mathrm{Ma}_2$, $\mathrm{Ra}_1 = (\nu\kappa\zeta\,\mathrm{Ra}_2)/\alpha$, $\mathrm{Pr}_1 = (\kappa\,\mathrm{Pr}_2)/\nu$ and $\mathrm{Pm}_1 = (\kappa\,\mathrm{Pm}_2)/\eta$ for given nondimensional ratios.

The linear stability of the basic state is analyzed in the usual way by seeking a solution for any physical quantity $\Phi(x, y, z, t)$ in normal mode form

$$\Phi(x, y, z, t) = \Phi_0(z) + \acute{\Phi}(z)\,\mathrm{e}^{(pt + ia_x + ia_y y)}, \tag{21}$$

where Φ_0 is the value of Φ in the basic state. In general, the temporal exponent p is complex; a_x and a_y are the wave numbers in the x and y directions, respectively. We have taken $\acute{u}_i$, $\acute{v}_i$ and $\acute{w}_i$ for the components of the perturbed velocities and $\acute{h}_{ix}$, $\acute{h}_{iy}$ and $\acute{h}_{iz}$ for the components of the perturbed magnetic fields in the convective motion.

Substituting these forms in the governing equations and the boundary conditions, we obtain linearized equations, neglecting the products and the squares of the perturbations. After eliminating $\acute{u}_i(z)$, $\acute{v}_i(z)$, $\acute{h}_{ix}(z)$ and $\acute{h}_{iy}(z)$ from the linearized forms of (3) to (5), using the linearized forms of (6) and (7), we have the following equations from which to determine the stability of the two-layer system.

$$(D^2 - a^2)\left[\left(D^2 - a^2 - \frac{\kappa p}{\mathrm{Pr}_1}\right)\acute{w}_1 + \frac{Q_1\,\mathrm{Pm}_1}{\kappa} D\acute{h}_{1z}\right] - \frac{\mathrm{Ra}_1 \beta a^2}{\kappa}\acute{T}_1 = 0, \tag{22}$$

$$(D^2 - a^2)\left[\left(D^2 - a^2 - \frac{p}{\mathrm{Pr}_2}\right)\acute{w}_2 + Q_2\,\mathrm{Pm}_2 D\acute{h}_{2z}\right] - \mathrm{Ra}_2 a^2 \acute{T}_2 = 0, \tag{23}$$

$$(\mathrm{Pm}_1(D^2 - a^2) - \kappa p)\acute{h}_{1z} + \kappa D\acute{w}_1 = 0, \tag{24}$$

$$(\mathrm{Pm}_2(D^2 - a^2) - p)\acute{h}_{2z} + D\acute{w}_2 = 0. \tag{25}$$

$$(D^2 - a^2 - \kappa p)\acute{T}_1 + \kappa\zeta\acute{w}_1 = 0, \tag{26}$$

$$(D^2 - a^2 - p)\acute{T}_2 + \acute{w}_2 = 0, \tag{27}$$

where $a = (a_x^2 + a_y^2)^{1/2}$, the total wave number in the (x, y) plane and $D = \mathrm{d}/\mathrm{d}z$, differentiation with respect to z. Combining the first two linearized equations derived from (3), using the continuity equations for the velocities and the magnetic fields, we obtain an expression for the fluid pressure

$$\acute{P}_2 = \frac{(\mathrm{Pr}_2(D^2 - a^2) - p)D\acute{w}_2 + Q_2\,\mathrm{Pr}_2\,\mathrm{Pm}_2(D^2 - a^2)\acute{h}_{2z}}{a^2}. \tag{28}$$

The expressions for external magnetic fields $\acute{\phi}_i$, which satisfy $(D^2 - a^2)\acute{\phi}_i = 0$, and $\acute{\phi}_i = 0$ at $z = \pm\infty$, are given by

$$\acute{\phi}_1(z) = B_1\,\mathrm{e}^{a(z+d)}, \qquad \acute{\phi}_2 = B_2\,\mathrm{e}^{a(1-z)}, \tag{29}$$

where B_i are arbitrary complex constants.

The corresponding boundary conditions are

$$D\acute{w}_1 = \acute{w}_1 = \acute{T}_1 = 0, \tag{30}$$

$$(D - a)\acute{h}_{1z} = 0, \tag{31}$$

at $z = -d$,

$$\acute{w}_1 = \acute{w}_2 = 0,\ D\acute{w}_1 = D\acute{w}_2,\ \acute{T}_1 = \acute{T}_2,\ D\acute{T}_1 = \zeta D\acute{T}_2, \tag{32}$$

$$\kappa(D^2 + a^2)\acute{w}_1 - \mu\kappa(D^2 + a^2)\acute{w}_2 + a^2\,\mathrm{Ma}_1\beta\acute{T}_1 = 0, \tag{33}$$

$$\acute{h}_{1z} = \acute{h}_{2z},\ D\acute{h}_{1z} = D\acute{h}_{2z} \tag{34}$$

at $z = 0$ and

$$\acute{w}_2 = p\acute{f}, \tag{35}$$

$$(D^2 + a^2)\acute{w}_2 + a^2\,\mathrm{Ma}_2(\acute{T}_2 - \acute{f}) = 0, \tag{36}$$

$$-\mathrm{Cr}(\mathrm{Pr}_2(D^2 - 3a^2 - Q_2) - p)D\acute{w}_2 - \mathrm{Cr}\,Q_2\,\mathrm{Pr}_2\,p\acute{h}_{2z} + \mathrm{Pr}_2 a^2(a^2 + \mathrm{Bo})\acute{f} = 0, \tag{37}$$

$$D\acute{T}_2 + \mathrm{Bi}(\acute{T}_2 - \acute{f}) = 0, \tag{38}$$

$$(D + a)\acute{h}_{2z} = 0 \tag{39}$$

at $z = 1$. Equation (37) is deduced from (15) by use of the value of $\acute{P}_2$ and (25). We obtain the convection problem for a single layer as studied by Wilson [12] for finite Bi from our analysis by taking $\zeta = 0$ and $d = 0$.

6. Solution of the linearized equations

The complete solution of the linear stability problem is determined, once we have solved (22) to (27) subject to the boundary conditions given in (30) to (39). The parameter p is the eigenvalue associated with a particular disturbance. If $\mathfrak{Re}(p) > 0$, the associated disturbance grows and the initial state is linearly unstable to that disturbance; if $\mathfrak{Re}(p) < 0$, the disturbance decays and the initial state is linearly stable. Disturbances with $\mathfrak{Re}(p) = 0$ are marginally stable. In the marginally stable state $\mathfrak{Im}(p)$ need not be zero and so oscillatory disturbances may exist. Exchange of stabilities has been proved to be valid for Bénard convection subject to a variety of boundary conditions by Pellew and Southwell [20] and for Marangoni convection in one fluid by Vidal and Acrivos [21]. Many of the investigations are concerned with the steady convection, but the first to investigate the possibility of an instability setting in an oscillatory convection was Takashima [6]. Later Wilson [11, 22] extended Takashima's [6] analysis by applying magnetic field and found overstability when the free surface is deformable and layer is cooled from below (negative Marangoni number). However, Sternling and Scriven [23] found both the stationary and the oscillatory marginal states for a two-fluid concentration dependent Marangoni convection model. Chandrasekhar [7], Kaddame and Lebon [24] have shown that exchange of stabilities does not hold for pure Bénard and pure Marangoni problems respectively. We have assumed that the principle of exchange of stabilities is valid for the present problem. For the sake of simplicity we consider only the case $\mathfrak{Im}(p) = 0$. If $\mathfrak{Im}(p) = 0$ assumed and instability is found, the apparent critical Marangoni number for steady convection must be an upper bound on the true critical Marangoni number. In the present work we shall assume the exchange of stabilities and so set $p = 0$ at the onset of convection. Eliminating $\acute{h}_{iz}$ from the coupled equations, we obtain for $\acute{w}_i$ and $\acute{T}_i$ as

$$\kappa((D^2 - a^2)^2 - Q_1 D^2)\acute{w}_1 = \mathrm{Ra}_1\beta a^2\acute{T}_1, \tag{40}$$

$$((D^2 - a^2)^2 - Q_2 D^2)\acute{w}_2 = \mathrm{Ra}_2 a^2\acute{T}_2. \tag{41}$$

Equations (26) and (27) reduce to

$$(D^2 - a^2)\acute{T}_1 + \kappa\zeta\,\acute{w}_1 = 0, \tag{42}$$

$$(D^2 - a^2)\acute{T}_2 + \acute{w}_2 = 0. \tag{43}$$

The linearized problem for the onset of steady Bénard-Marangoni convection is solved by seeking solutions of the form

$$\acute{w}_i(z) = A_i C_i \,\mathrm{e}^{\xi_i z}, \qquad \acute{T}_i(z) = C_i \,\mathrm{e}^{\xi_i z},$$

where exponents ξ_i, coefficients A_i and C_i are to be determined. Substituting these forms in (40)–(43) and eliminating A_i and C_i, we have

$$(\xi_i^2 - a^2)((\xi_i^2 - a^2)^2 - Q_i \xi_i^2) + \mathrm{Ra}_i a^2 = 0, \tag{44}$$

which gives six distinct roots ξ_{ij} with $j = 1$ to 6. Denoting the values of A_i and C_i corresponding to ξ_{ij} by A_{ij} and C_{ij}, from (42) and (43), we get

$$A_{1j} = -\frac{\beta}{\kappa}(\xi_{1j}^2 - a^2), \qquad A_{2j} = -(\xi_{2j}^2 - a^2). \tag{45}$$

The general solution to the linear stability problem is therefore

$$\acute{w}_i(z) = \sum_{j=1}^{6} A_{ij} C_{ij} \,\mathrm{e}^{\xi_{ij} z}, \qquad \acute{T}_i(z) = \sum_{j=1}^{6} C_{ij} \,\mathrm{e}^{\xi_{ij} z}. \tag{46}$$

From the boundary conditions (35) and (37) with $p = 0$, we get the free-surface deflection evaluated at $z = 1$ as

$$\acute{f} = \mathrm{Cr} \frac{(D^2 - 3a^2 - Q_2) D \acute{w}_2}{a^2(a^2 + \mathrm{Bo})}. \tag{47}$$

Omitting the magnetic field boundary conditions given in (31), (34) and (39), we are left with twelve boundary conditions given in (30), (32), (33), (35), (36) and (38) to determine the twelve unknowns $C_{ij}, i = 1, 2, \; j = 1$ to 6 (up to an arbitrary multiplier). Substitution of above expressions for $\acute{w}_i(z)$ and $\acute{T}_i(z)$, given in (46), in the twelve boundary conditions gives rise to a 12×12 complex determinant of coefficients of unknowns C_{ij}, which after some simplification can be written in the form

$$\kappa D_1 + a^2 \beta \,\mathrm{Ma}_1 D_2 + \kappa a^2 \,\mathrm{Ma}_2 D_3 + a^4 \beta \,\mathrm{Ma}_2 \,\mathrm{Ma}_1 D_4 = 0. \tag{48}$$

The dispersion relation for marginal stability (48) depends on all the nondimensional parameters except Pr_i, Pm_i and is quadratic in Marangoni number, Ma_1 or Ma_2. The four 12×12 complex determinants D_r, $r = 1, 2, 3, 4$ depend on all the other parameters of the problem except Ma_1 and Ma_2. The elements of determinants $D_r = |d_{l,m}^r|$ can be obtained in a straight forward way and they are not given here as that adds to the length of the paper.

Here it is observed that D_1 and D_2 are independent of Cr and Bo when $\mathrm{Bi} = 0$ and, D_3 and D_4 are independent of Bi. When $\mathrm{Bi} = 0$ and the Marangoni numbers are zero, (48) reduces to $D_1 = 0$, which implies the onset of steady Bénard convection is independent of surface tension of the free surface when fluid-fluid interface is flat, and is analogous to the single layer case. Wilson [19] has pointed out that in agreement with the present results the qualifier 'when $Nu = 0$' should be added to the line 'one immediate consequence of this is that the onset of steady Bénard convection ($M = 0$) is independent of Cr and Bo' in [12]. The complete solution of the linear stability problem is given by (44) and (48). Equation (44) is solved for ξ_{ij} using Numerical Algorithms Group (NAG) routine CO2AGF. In order to prevent numerical difficulties arising from the exponential terms present in the determinants D_r, we multiplied each column m when $m \leqslant 6$ by an exponential factor with exponent min(0,$-\Re\mathfrak{e}(-\mathrm{d}\xi_{1m})$) and when $m \geqslant 7$ by min(0, $-\Re\mathfrak{e}(\xi_{2n})$) where $\Re\mathfrak{e}(.)$ denotes the real part of a complex quantity. The complex valued determinants D_1, D_2, D_3 and D_4 are evaluated numerically by using NAG routine FO3ADF. Equation (48) is solved for Marangoni number by use of NAG routine CO2AFF.

7. Discussion of results

The marginal curves in the (a, Ma_2) plane are obtained by (48) where Ma_2 is a function of the parameters a, Ra_2, Cr, Q_2, Bo, Bi, d, λ, ρ, μ, ζ, α, η, $\bar{\mu}$ and κ. There are two values for Ma_2, one of them being always positive and the other is either positive or negative giving rise to two marginal curves. The curve with negative values of the Marangoni number corresponds to a system heated from above and therefore is not possible for a system heated from below. When both values of Ma_2 are positive, both marginal curves possess global minima. For a given set of parameters the critical Marangoni number for the onset of steady convection is defined as the minimum of the global minima of both marginal curves. We denote this critical value by Ma_{2c} and the corresponding critical wave number by a_c. All disturbances with $\mathrm{Ma}_2 < \mathrm{Ma}_{2c}$ are stable and there exist unstable disturbances for $\mathrm{Ma}_2 > \mathrm{Ma}_{2c}$. The Bénard convection (48) becomes a transcendental equation in Ra_2 when the other parameters are prescribed. Similarly, the critical Rayleigh number Ra_{2c} for a given set of parameters is defined as the minimum of global minima of each marginal curve in the (a, Ra_2) plane. Here also, the region above the marginal stability curves represent unstable modes and the region below the curves represents stable modes. We present here the critical Rayleigh number Ra_{2c}, the critical Marangoni number Ma_{2c} for fluid 2 only as one can trivially calculate the corresponding critical parameters for fluid 1. In the numerical calculations following Doi and Koster [17], we have taken $\mu = 10$, $\zeta = 1$, $\kappa = 0{\cdot}1$, $\eta = 10{\cdot}0$, $\rho = 1{\cdot}0$, $\lambda = 2$, $\alpha = 3{\cdot}0$ and $\bar{\mu} = 0{\cdot}1$.

7.1. NONDEFORMABLE FREE SURFACE (Cr = 0)

When Cr $= 0$ the free surface is nondeformable. In this case the problem is independent of Bo and the critical Marangoni number Ma_2 and the corresponding wave number a depend on Q_2, Ra_2 and Bi. Similarly, the critical Rayleigh number and the corresponding wave number depend on Q_2, Ma_2 and Bi.

*The free surface is partially insulated (*Bi $< \infty$*)*
Numerically calculated values of Ma_{2c} and the corresponding values of a_c are given in Table 2 for pure Marangoni convection ($\mathrm{Ra}_2 = 0$) with $d = 1$ and $d = 0$ for a range of values of Q_2 when the free surface is perfectly insulated (Bi $= 0$).

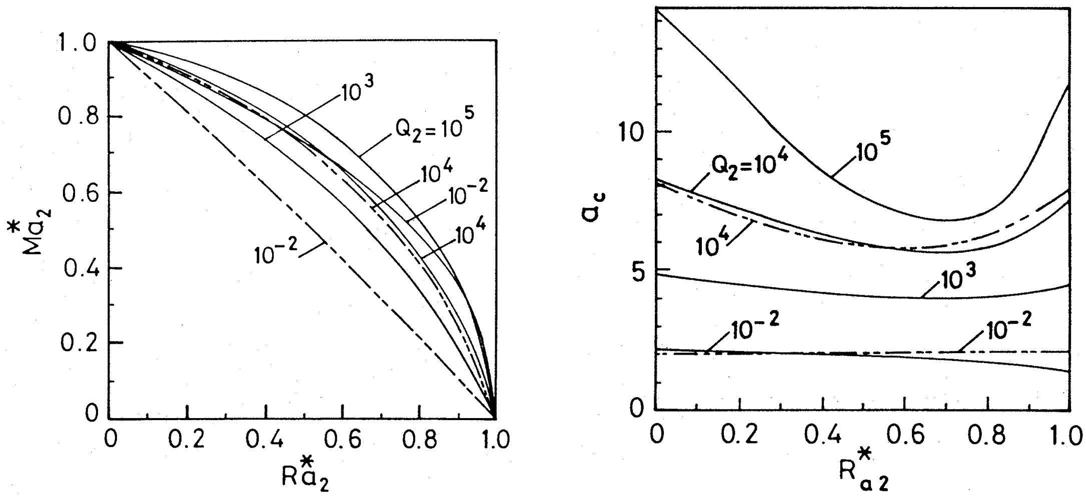
The effect of the magnetic field for two layer system remain the same as for a single layer, namely a monotonic increase in Ma_{2c} and a_c viewed as a function of Q_2. The critical values of Ra_{2c} and the corresponding values of a_c for pure Bénard convection ($\mathrm{Ma}_2 = 0$) with $d = 1$ and $d = 0$ for different values of Q_2 are given in Table 3. Here also the effect of the magnetic field remains the same, namely a monotonic increase in Ra_{2c} and a_c as a function of Q_2. It is observed that for a given Q_2, Ma_{2c} for $d = 1$ is always greater than that for $d = 0$ (Table 2)

Table 2. Critical Ma_{2c} and the corresponding a_c for pure Marangoni convection when $Cr = Bi = 0$ for different values of Q_2 with $d = 1$ and $d = 0$.

	$d = 1$		$d = 0$	
Q_2	a_c	Ma_{2c}	a_c	Ma_{2c}
0	2·179	95·694	1·992	79·607
10^{-4}	2·179	95·690	1·993	79·607
10^{-3}	2·178	95·654	1·993	79·609
10^{-2}	2·168	95·329	1·993	79·633
10^{-1}	2·124	93·733	1·995	79·864
10^{0}	2·038	91·716	2·015	82·172
10^{1}	2·105	109·176	2·181	104·223
10^{2}	2·946	291·375	2·959	284·222
10^{3}	4·869	1651·54	4·745	1632·47
10^{4}	8·271	12875·72	8·092	12830·16
10^{5}	14·36	114326·75	14·19	114212·7
10^{6}	25·28	1075639·1	25·12	1075322·1
10^{7}	44·76	10411114·5	44·60	10410179·7
10^{8}	79·44	102268999·0	79·28	102266154·0

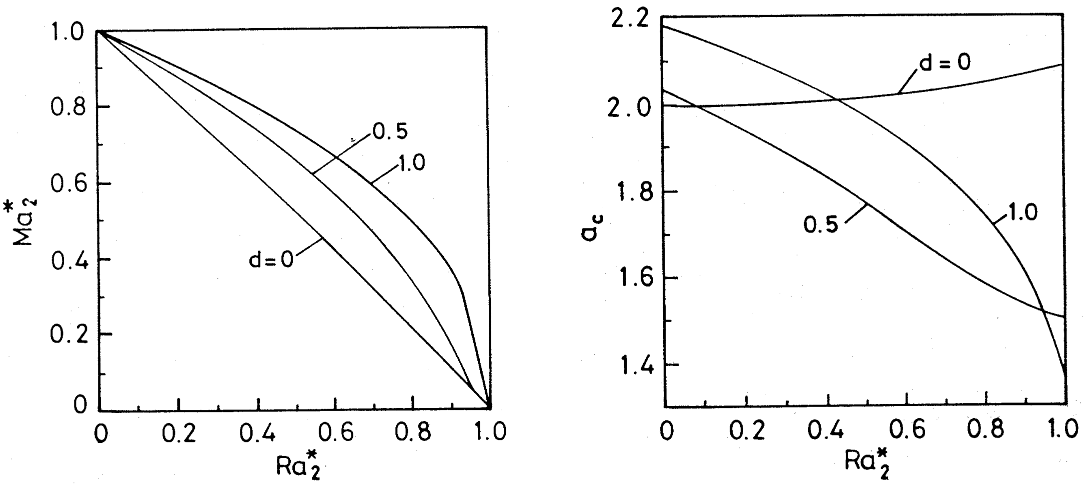
Table 3. Critical Ra_{2c} and the corresponding a_c for pure Marangoni convection when $Cr = Bi = 0$ for different values of Q_2 with $d = 1$ and $d = 0$.

	$d = 1$		$d = 0$	
Q_2	a_c	Ra_{2c}	a_c	Ra_{2c}
0	1·363	280·766	2·086	668·998
10^{-4}	1·363	280·792	2·086	669·000
10^{-3}	1·364	281·016	2·086	669·020
10^{-2}	1·368	283·119	2·086	669·213
10^{-1}	1·394	297·554	2·088	671·145
10^{0}	1·483	355·705	2·109	690·373
10^{1}	1·739	581·730	2·288	874·862
10^{2}	2·586	2039·27	3·128	2424·90
10^{3}	4·475	13723·76	4·991	14594·63
10^{4}	7·509	115981·9	7·949	118360·3
10^{5}	11·86	1067798·1	12·23	1074679·1
10^{6}	18·10	10249968·7	18·42	10270542·7
10^{7}	27·13	100484253·0	27·43	100547094·0
10^{8}	40·31	995347433·0	40·59	995541915·0



(a) (b)

Figure 2. Critical conditions for the onset of stationary convection in the case Cr = 0 and Bi = 0 plotted as functions of Ra_2^* for $Q_2 = 10^{-2}$, 10^3, 10^4 and 10^5 (a) Ma_2^* (b) a_c. Continuous lines for $d = 1$ and broken lines for $d = 0$.



(a) (b)

Figure 3. Critical conditions for the onset of stationary convection in the case Cr = 0, $Q_2 = 0$ and Bi = 0 plotted as functions of Ra_2^* for $d = 0$, 0·5 and 1 (a) Ma_2^* (b) a_c.

whereas Ra_{2c} for $d = 1$ is always less than that for $d = 0$ (Table 3). Thus, the effect of the presence of the lower fluid layer increases the range of stability for the pure Marangoni convection, whereas it is reduced for the pure Bénard convection with magnetic field. In both cases results for the case $d = 0$ (single layer) are in excellent agreement with those given in Table 1 of Wilson [12].

The values of Ma_2^* and a_c are plotted as a function of Ra_2^* for different values of Q_2 in Figure 2, where Ra_2^* is defined as the ratio of Ra_2 to the corresponding value of Ra_{2c} for pure Bénard convection and Ma_2^* is defined as the ratio of Ma_2 to the corresponding value of Ma_{2c} for pure Marangoni convection. From Figure 2(a) it is seen that for $d = 0$ the critical Marangoni number decreases with an increase of the Rayleigh number, approximately linearly for $Q_2 = 10^{-2}$ and nonlinearly for $Q_2 = 10^4$. Thus, for large Q_2 the two destabilizing agencies are tightly coupled and reinforce each other, confirming the results of Nield [8]. Further, as seen from Figure 2(a), the coupling is weakened as Q_2 is increased, as found by Wilson [12] in the single-layer case. However, when the magnetic forces are weak, the curve for $Q_2 = 10^{-2}$, $d = 1$ shows that the presence of the lower layer weakens the coupling of two destabilizing mechanisms. Figure 2(b) shows as Q_2 increases, the variation of a_c with Ra_2^* is greater. The critical Ma_2^* for the onset of steady convection with $\mathrm{Cr} = \mathrm{Bi} = Q_2 = 0$ for different d is depicted in Figure 3 which confirms the weakening of the coupling with an increase in d. The value of a_c with $d = 1$ for Ra_2^* near zero (Marangoni convection) is always greater than that with $d = 0$, whereas it is opposite for Ra_2^* near one (Bénard convection) as seen from Figure 3(b) for nonmagnetic case, *i.e.* with $Q_2 = 0$.

As the stability in the pure Marangoni convection is improved and, in the pure Bénard convection reduced for nonzero values of Q_2, here we analyse asymptotically the stability for the pure Marangoni convection for large Q_2. We observe from numerical results that the critical Marangoni number is of order Q_2 for large Q_2 and also motivated by the results of Wilson [12] for the single-layer problem we seek a solution in which $a = o(Q_2^{1/4})$. Substituting the asymptotic values of D_1, D_2, D_3 and D_4 for large Q_2 in (48) and solving for Ma_2, we get

$$\mathrm{Ma}_2^{(1)} = Q_2 + f_1^{(1)} Q_2^{3/4} + o(Q_2^{3/4}), \tag{49}$$

$$\mathrm{Ma}_2^{(2)} = f_0^{(2)} Q_2 + f_1^{(2)} Q_2^{3/4} + o(Q_2^{3/4}), \tag{50}$$

$$f_1^{(1)\,\mathrm{or}\,(2)} = \frac{1}{2\lambda\mu s^2}\left(\frac{E_{42}(E_2 \pm E_3)}{E_{41}^2} - \frac{2E_4E_3 \pm E_5}{2E_3E_{41}}\right), \qquad f_0^{(2)} = \frac{\zeta(1+\zeta)(\eta + \eta_0\bar{\mu})}{2\lambda\bar{\mu}(\kappa - \eta_0\zeta^2)},$$

where E_1, E_2, E_3, E_4 and E_5 can be determined, $+$ and $-$ correspond to $f_1^{(1)}$ and $f_1^{(2)}$, respectively.

The value of s in the wave number $a = sQ_2^{1/4} + o(Q_2^{1/4})$ is determined numerically by finding the root of $\mathrm{d}f_1^{(1)}/\mathrm{d}s = 0$. For $\mathrm{Bi} = 0$, we get $s = 0{\cdot}79428$ and the corresponding value of the Marangoni number calculated fram (49) with $Q_2 = 10^8, d = 1$ is $\mathrm{Ma}_{2c}^{(1)} = 102268709{\cdot}0$, and this is in good agreement with the corresponding numerical value given in Table 2. In the limit $d \to 0$ we recover the results of Wilson [12],

$$\mathrm{Ma}_2^{(1)} = Q_2 + \left(\frac{\mathrm{Bi}}{s} + \frac{2s}{1 - \mathrm{e}^{-2s^2}}\right) Q_2^{3/4} + o(Q_2^{3/4}). \tag{51}$$

Figure 4 shows the numerically calculated values of Ma_{2c}/Q_2 and $a_c/Q_2^{1/4}$ plotted as function of Q_2 for different values of Bi and verifies the values in the asymptotic limit.

Free surface is conduction ($\mathrm{Bi} = \infty$)

In this case results are independent of the presence of the lower fluid layer and exactly coincidence with the results for a single layer given by Wilson [12]. In the limit $Q_2 \to \infty$ Wilson [12] showed.

$$\frac{\mathrm{Ma}_2}{\mathrm{Bi}} = \frac{1}{s}\left(1 - \frac{2s}{\sqrt{(1 + 4s^2)}}\right)^{-1} Q_2^{1/2} + o(Q_2^{1/2}). \tag{52}$$

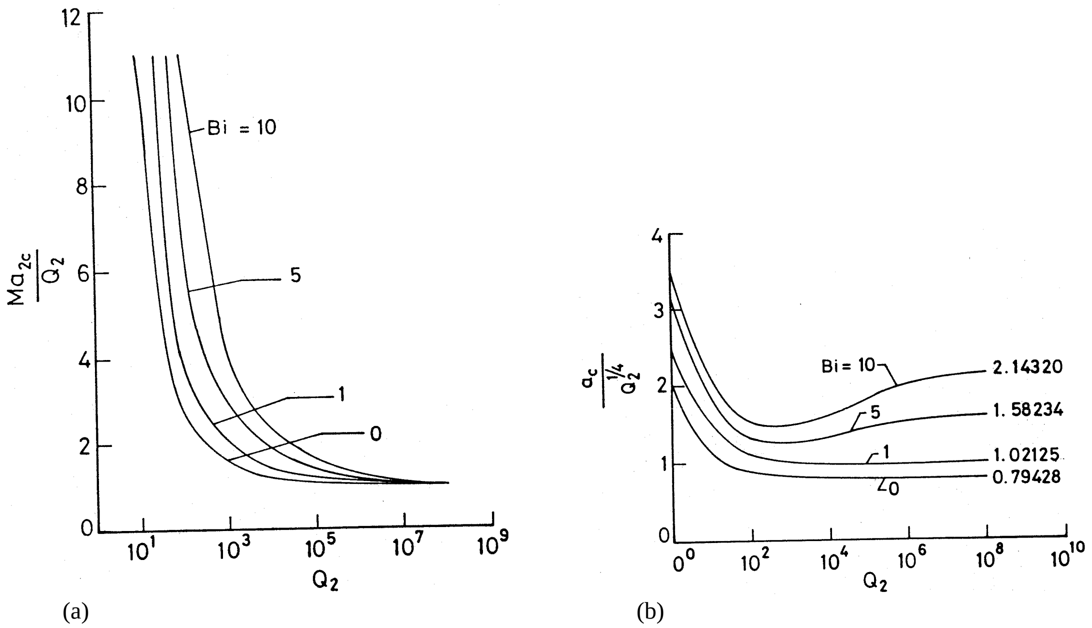


Figure 4. Comparison of numerically calculated and asymptotic results for pure marangoni convection in the limit $Q_2 \to \infty$ plotted as functions of Q_2 when Cr = 0 for Bi = 0, 1, 5 and 10 (a) Ma_{2c}/Q_2 (b) $a_c/Q_2^{1/4}$.

Table 4. Numerically calculated values of M_{2c}, Ra_{2c} and the corresponding values of a_c for pure Marangoni and Bénard convection, when Cr = 0 and Bi = 0 in the case $Q_2 = 100$ for a range of values of d when $\lambda = 2$.

d	a_c	Ma_{2c}	a_c	Ra_{2c}
1·0	2·946	291·375	2·586	2039·268
0·9	2·946	291·351	2·595	2040·427
0·8	2·944	291·310	2·606	2042·335
0·7	2·943	291·239	2·624	2045·467
0·6	2·939	291·112	2·648	2050·592
0·5	2·936	290·886	2·683	2058·970
0·4	2·929	290·477	2·731	2072·685
0·3	2·923	289·732	2·793	2095·315
0·2	2·915	288·351	2·876	2133·312
0·1	2·914	285·727	2·982	2199·498
10^{-2}	2·938	282·059	3·104	2333·485
10^{-3}	2·956	283·798	3·125	2411·306
10^{-4}	2·959	284·177	3·128	2423·502
10^{-5}	2·959	284·218	3·128	2424·763

Table 5. Numerically calculated values of M_{2c} and the corresponding values of a_c for pure Marangoni convection when Cr = 0 and Bi = 0 in the case $Q_2 = 100$ for a range of values of λ when $d = 1$.

λ	a_c	Ma_{2c}	λ	a_c	Ma_{2c}
1·0	2·822	282·474	0·3	2·720	274·956
0·9	2·808	281·469	0·2	2·705	273·788
0·8	2·794	280·441	0·1	2·689	271·595
0·7	2·780	279·391	10^{-2}	2·675	271·501
0·6	2·765	278·318	10^{-3}	2·673	271·390
0·5	2·750	277·221	10^{-4}	2·673	271·379
0·4	2·735	276·101	10^{-5}	2·673	271·378

Effect of λ and d

For Cr = 0, Bo = 1, Bi = 0, $Q_2 = 100$ numerically calculated values of Ma_{2c}, Ra_{2c} and corresponding wave numbers when $\lambda = 2$ for different values of d are given in Table 4. The results in Table 4 confirm that, in the limit $d \to 0$, numerically calculated results approach the results of Wilson [12]. In Table 5 numerically calculated values of Ma_{2c}, a_c when Cr = 0, Bo = 1, Bi = 0, $Q_2 = 100$, $d = 1$ for different λ are given and from which it can be concluded that λ has a stabilizing effect on the two-layer system.

7.2. LONG WAVELENGTH ASYMPTOTICS WITH DEFORMABLE FREE SURFACE (Cr $\neq$ 0)

The long-wavelength asymptotics with deformable free surface are calculated with MATHEMATICA. The behaviour of the marginal stability curves when Cr $\neq$ 0 depends on Bi and Bo, just as it does in the absence of the magnetic field. There are two different cases of interest given by Bo = 0 and Bo $\neq$ 0. In order to determine this, we seek the expansion of Ma_2 in the long-wavelength disturbance about $a = 0$ in powers of a^2 by writing $Ma_2 = M_{-2}a^{-2} + M_0 + M_2a^2 + O(a^4)$ on the marginal stability curves. If $M_{-2} > 0$ then the system is stable for long-wavelength disturbances and if $M_{-2} < 0$ then the system is unstable. If $M_{-2} = 0$ and $M_0 > 0$, then the marginal stability curves for long-wavelength disturbances will have positive local maximum or minimum at $a = 0$ according as $M_2 < 0$ or $M_2 > 0$. The global minimum is at nonzero a for $M_2 < 0$ and at $a = 0$ for $M_2 > 0$. If $M_{-2} = 0$ and $M_0 < 0$, then the marginal curve for long wavelength disturbances will have negative local maximum or minimum according to $M_2 > 0$ or $M_2 < 0$.

(i) Bo = 0.

When Bo = 0 the marginal stability curves are analysed for the onset of steady pure Marangoni convection in the (a, Ma_2) plane in the limit $a \to 0$. When $Q_i \neq 0$, Bo = 0 and all other parameters in the problem apart from Ma_2 are fixed, the roots of (44) ξ_{ij}, A_{ij} and exponential terms in the elements of the determinants are expanded in powers of a^2 up to

$O(a^{15})$. Using these expanded forms, we observe that the determinants D_1, D_3, D_4, appearing in (48) are given by

$$\begin{aligned} D_1 &= D_{11}a^{14} + O(a^{15}), \\ D_2 &= D_{21}a^{10} + O(a^{15}), \\ D_3 &= D_{31}a^{10} + O(a^{15}), \\ D_4 &= D_{41}a^{10} + O(a^{15}), \end{aligned} \tag{53}$$

where

$$\begin{aligned} D_{11} &= (64Q_1^2Q_2^{5/2}/\kappa^4\zeta^4)(1+\mathrm{Bi}+d\,\mathrm{Bi}\zeta) \\ &\quad \times(Q_1^{1/2}sq_1(sq_2-Q_2^{1/2}cq_2)+dQ_1cq_1(Q_2^{1/2}cq_2-sq_2) \\ &\quad +\mu Q_2 sq_2(dQ^{1/2}sq_1-2cq_1+2)), \\ D_{21} &= \mathrm{Cr}(64d\,\mathrm{Bi}Q_1^2Q_2^{7/2}/\kappa^4\zeta^3)sq_2(dQ_1^{1/2}sq_1-2cq_1+2), \\ D_{31} &= \mathrm{Cr}(64Q_1^2Q_2^3/\kappa^4\zeta^4)(Q_1^{1/2}(cq_2-1)(sq_1-dQ_1^{1/2}cq_1) \\ &\qquad +\mu Q_2^{1/2}sq_2(2cq_1-2-dQ_1^{1/2}sq_1)), \\ D_{41} &= \mathrm{Cr}(32dQ_1Q_2^2/3\kappa^4\zeta^3)((1+2cq_1)(-1+cq_2)d^2\kappa Q_1Q_2 \\ &\qquad +12(-1+cq_2)Q_1(2-2cq_1+dQ_1^{1/2}sq_1) \\ &\quad +3(1-cq_2)\kappa Q_2(4-4cq_1+3dQ_1^{1/2}sq_1) \\ &\quad +6Q_1Q_2^{1/2}(-2+2cq_1-dQ_1^{1/2}sq_1)sq_2), \end{aligned}$$

$$cq_1=\cosh(dQ_1^{1/2}),\quad cq_2=\cosh(Q_2^{1/2}),\quad sq_1=\sinh(dQ_1^{1/2})\quad\text{and}\quad sq_2=\sinh(Q_2^{1/2}).$$

The roots of (48) are given by

$$\mathrm{Ma}_2^{(1)} = -\frac{(\lambda\mu D_{21}+D_{31})}{\lambda\mu D_{41}}\frac{1}{a^2}+O(1), \tag{54}$$

$$\mathrm{Ma}_2^{(2)} = -\frac{(D_{11}}{(\lambda\mu D_{21}+D_{31})}a^2+O(a^4). \tag{55}$$

The sign of the coefficients of (54) and (55) depends on Q_2 and d. The coefficient of a^{-2} in (54) is positive for $Q_2 < 1$ and negative for $Q_2 \geqslant 1$. The marginal curve corresponding to (54) gives nonzero minimum Marangoni number for nonzero wave number, but the marginal curve corresponding to (55) gives zero minimum Marangoni number at $a = 0$ as the leading order term of (55) is zero. The marginal stability curve attains a positive minimum value zero at $a = 0$. Hence, when $\mathrm{Bo} = 0$, the system is always unstable. In the limit $d \to 0$ (55) reduces to

$$\mathrm{Ma}_2^{(2)} = \frac{(1+\mathrm{Bi})}{\mathrm{Cr}}G_1a^2+O(a^4), \tag{56}$$

where

$$G_1 = \frac{Q_2^{1/2} cq_2 - sq_2}{Q_2^{1/2}(cq_2 - 1)}.$$

Equation (56) was first obtained by Wilson [11]. Since, there is no term independent of a in (56), when $\mathrm{Bo} = 0$, the system is always unstable as the local (hence global) minimum zero exists at $a = 0$ in this case.

(ii) $\mathrm{Bo} \neq 0$ then

When $\mathrm{Bo} \neq 0$ then

$$\mathrm{Ma}_2^{(1)} = -\frac{(\lambda\mu D_{21} + D_{31})}{\lambda\mu D_{41}} \frac{1}{a^2} + O(1) \tag{57}$$

$$\mathrm{Ma}_2^{(2)} = -\frac{\mathrm{Bo} D_{11}}{(\lambda\mu D_{21} + D_{31})} + O(a^2). \tag{58}$$

The coefficient of a^{-2} in (57) is positive for $Q_2 < 1$ and negative for $Q_2 \geqslant 1$. Similarly, the leading-order term of (58) is negative when $Q_2 < 1$ and positive when $Q_2 \geqslant 1$. The marginal curve corresponding to (57) gives the critical Marangoni number for nonzero wave number for $Q_2 < 1$ and it is checked numerically that the coefficient of a^2 of (58) is negative when $Q_2 \geqslant 1$ and therefore for $Q_2 \geqslant 1$ the marginal curve corresponding to (58) gives a nonzero critical Marangoni number for a nonzero wave number. In this case only long-wavelength disturbances are stable with moderate or weak magnetic field beacuse the global minimum for the marginal stability curve exists for nonzero a. In the limit $d \to 0$, (58) reduces to

$$\begin{aligned} \mathrm{Ma}_2^{(2)} \;=\; & \frac{\mathrm{Bo}(1+\mathrm{Bi})}{\mathrm{Cr}} G_1 + \left[\frac{\mathrm{Bo}}{2\,\mathrm{Cr} Q_2}(2G_1 + G_2) + \frac{\mathrm{Bo\,Bi}}{6\,\mathrm{Cr} Q_2}(6G_1 + G_3) \right. \\ & \left. - \frac{(1+\mathrm{Bi})}{6\,\mathrm{Cr}^2 Q_2^2} G_1 (\mathrm{Bo}^2 G_4 + 3\,\mathrm{Cr} Q_2 G_5) \right] a^2 + O(a^4), \end{aligned} \tag{59}$$

where

$$G_2 = \frac{Q_2^{1/2}(Q_2^{1/2} cq_2 + sq_2)}{(cq_2 - 1)},$$

$$G_3 = \frac{Q_2^{1/2}(Q_2^{1/2} cq_2 + 5 + sq_2)}{(cq_2 - 1)},$$

$$G_4 = \frac{12(cq_2 - 1) - Q_2^{1/2}(9sq_2 - 2Q_2^{1/2} cq_2 - Q_2^{1/2})}{(cq_2 - 1)},$$

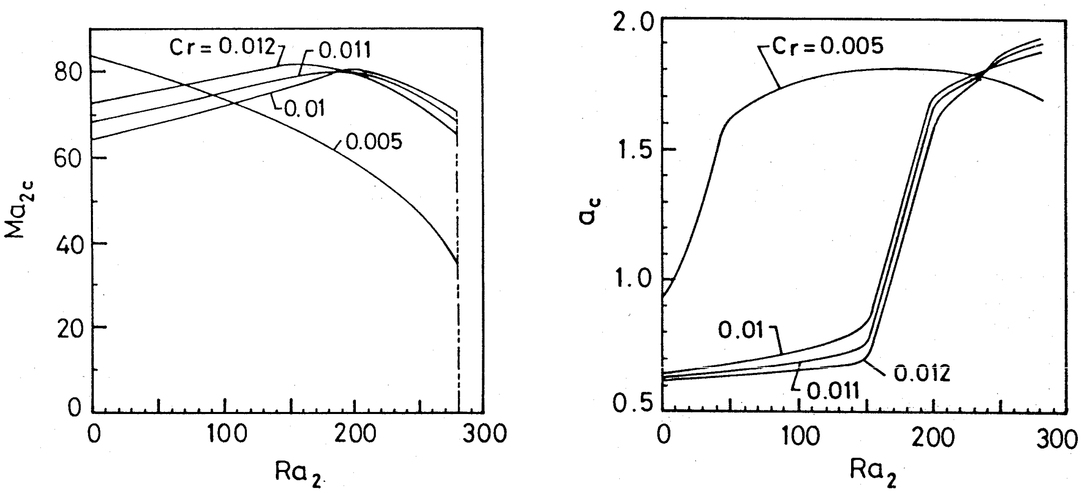
$$G_5 = \mathrm{Bo} Q_2 + 2(4\,\mathrm{Bo} - Q_2).$$

The coefficient of a^2 in (59) is negative when

$$\mathrm{Cr} < \mathrm{Cr}^* = \frac{\mathrm{Bo}^2(1+\mathrm{Bi}) G_1 G_4}{Q_2(3\,\mathrm{Bo}(2G_1 + G_2) + \mathrm{Bo\ Bi}(6G_1 + G_3) - 3(1+\mathrm{Bi}) G_1 G_5)}. \tag{60}$$

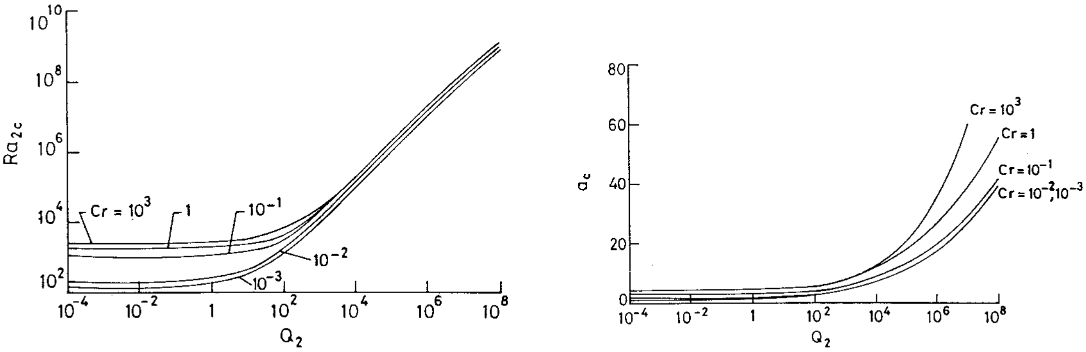
Table 6. Value of Cr* for different Q_2 when Bi = 1 and Bo = 1.

Q_2	Cr*	Q_2	Cr*	Q_2	Cr*
10^{-4}	$8{\cdot}06 \times 10^{-3}$	10^{0}	$7{\cdot}86 \times 10^{-3}$	10^{4}	$3{\cdot}78 \times 10^{-5}$
10^{-3}	$8{\cdot}06 \times 10^{-3}$	10^{1}	$6{\cdot}36 \times 10^{-3}$	10^{5}	$3{\cdot}93 \times 10^{-6}$
10^{-2}	$8{\cdot}06 \times 10^{-3}$	10^{2}	$2{\cdot}22 \times 10^{-3}$	10^{6}	$3{\cdot}98 \times 10^{-7}$
10^{-1}	$8{\cdot}04 \times 10^{-3}$	10^{3}	$3{\cdot}34 \times 10^{-4}$	10^{7}	$3{\cdot}99 \times 10^{-8}$



(a) (b)

Figure 5. Numerically calculated values of (a) Ma_{2c} and (b) a_c in the case $Q_2 = 0$, Bo = 1, Bi = 0 plotted as functions of Ra_2 for a range of values of Cr = 0·005, 0·01, 0·011 and 0·012.



(a) (b)

Figure 6. Numerically calculated values of (a) Ra_{2c} and (b) a_c in the case $Ma_2 = 25$, Bo = 1, Bi = 0 plotted as functions of Q_2 for a range of values of Cr.

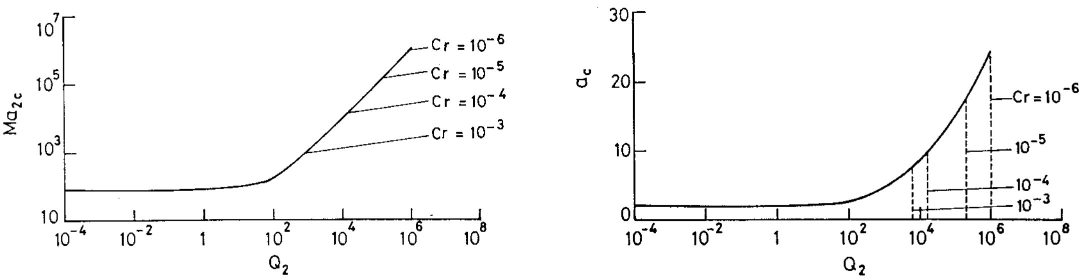


Figure 7. Numerically calculated values of (a) Ma_{2c} and (b) a_c in the case $\mathrm{Ra}_2 = 0$, $\mathrm{Bo} = 1$, $\mathrm{Bi} = 0$ plotted as functions of Q_2 for a range of values of Cr.

In the limit $Q_2 \to 0$ in (60), we recover the well-known result of Takashima [5] which is

$$\mathrm{Cr} < \mathrm{Cr}^* = \frac{(1+\mathrm{Bi})\mathrm{Bo}^2}{8(1+\mathrm{Bi})(15-2\,\mathrm{Bo})+40\,\mathrm{Bo}}. \tag{61}$$

Evidently, as Q_2 increases the critical Cr^* reduces. This implies that for a fixed $\mathrm{Cr} < \mathrm{Cr}^*$, a certain disturbance could be stabilized by choosing a suitable magnetic field. When the magnetic field is large, even for very small value of Cr, *i.e.* for a small deformation of the free surface, the system is unstable. Comparing the value of Cr^* for different values Q_2 given in Table 6, and corresponding Cr^* when $Q_2 = 0$, it is concluded that the magnetic field has no stabilizing effect on the system when $\mathrm{Cr} \neq 0$ and $\mathrm{Bo} \neq 0$.

Critical Ma_{2c} and a_c are plotted as function of Ra_2 with $Q_2 = 0$, $\mathrm{Bo} = 1$, $d = 1$ and $\mathrm{Bi} = 0$ for different values of Cr in Figure 5 and it shows the variation in the values of Ma_{2c} and a_c as Cr increases compared to the case with $d = 0$. Figure 5(a) depicts a jump discontinuity in Ma_{2c} and it goes to zero for Ra_2 around 280. Another interesting feature is that for a double-layer system a_c is not zero for any Ra_2, unlike a single layer system for Cr lying in (0·005, 0·01) as seen in Figure 5(b). In Figure 6 typical values of Ra_{2c} and a_c are plotted as a function of Q_2 for the case with $\mathrm{Bo} = 1$, $\mathrm{Ma}_2 = 25$, $d = 1$ and $\mathrm{Bi} = 0$ and it shows that they only differ significantly from those when $\mathrm{Cr} = 0$ for unrealistically large values of Cr. The variation of Ma_{2c} and a_c with $\mathrm{Ra}_2 = 0$, $\mathrm{Bo} = 1$, $d = 1$ and $\mathrm{Bi} = 0$ when the free surface is deformable are given in Figure 7 and it is concluded that the lower layer has no stabilizing effect on the system for deformable free surface, whereas it stabilizes the system for a non-deformable free surface.

8. Conclusions

In this paper a combination of analytical and numerical techniques have been used to analyse the effect of a second layer and a magnetic field on the onset of steady Bénard-Marangoni convection in a two-layer system of conducting fluids subjected to a uniform vertical temperature gradient. It is found that the parameters Cr, Bo and λ play an important role on the onset of steady convection. In the presence of a second layer it is observed that the critical parameters for the onset of pure Marangoni convection are increased, whereas for the onset of pure bouyancy convection they are decreased. Further, we have discussed in detail the behaviour of the critical parameters Ra_{2c}, Ma_{2c} and a_c in different limiting cases $Q_2 \to 0$ and $d \to 0$.

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